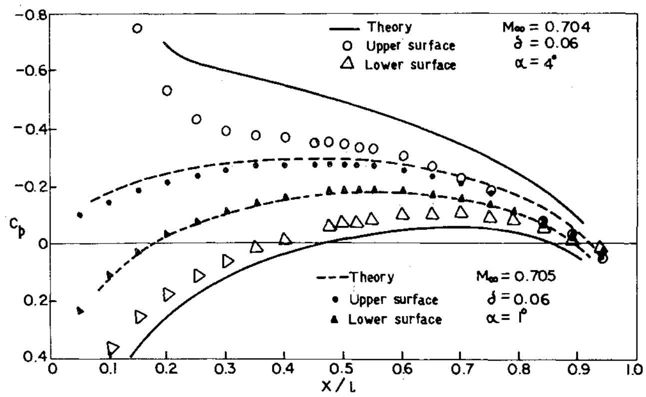

Fig. 4 Pressure distribution on circular arc airfoils at incidence at transonic speed.

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Isoperimetric Formulation for Some Problems of Optimization of the Entry into Atmosphere

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Nomenclature

z = altitude
x = distance upon horizontal
ρ = density $= \rho_0 e^{-\beta z}$
S = reference surface
S_v = area of the vehicle surface
V = velocity of the vehicle
C_z = lift coefficient
C_x = drag coefficient
θ = inclination angle of the trajectory with respect to the local horizontal
m = vehicle mass

Introduction

THE optimization of the entry of space vehicles into the planetary atmosphere constitutes at the present time the object of many preoccupations. In this respect author's paper[1] gives a formulation that may be generalized to several entry problems.

In principle we refer to the problems of the minimum time lifting entry, the lifting entry with minimum total heat yielded to the vehicle, the lifting entry with minimum consumption of ablative mass and the lifting entry with minimum heat yielded in the critical zone. The present Note approaches briefly two problems, the others being only mentioned.

In all cases it is assumed that the entry is made on a plane trajectory at small initial angles and the lift differs from zero, the angle θ being considered positive in the descent.

The governing motion equations are

$$dV/dt = -(SC_x/2m)\rho V^2; \quad dz/dt = -V\sin\theta$$
$$d\theta/dt = -(SC_z/2m)\rho V; \quad dx/dt = V\cos\theta \tag{1}$$

Minimum Time Optimal Lifting Entry into Planetary Atmosphere

In the following we determine the optimal laws of variation of the motion parameters such that a lifting vehicle should travel the descending trajectory between the altitudes z_i, z_f in the shortest time, the distance on the horizontal being given.

We introduce the variable z such that the two first equations (1) become

$$dV/dz = (SC_x/2m)\,\rho V/\sin\theta$$
$$d\theta/dz = (SC_z/2m)\,\rho/\sin\theta \tag{2}$$

From the first Eq. (2), denoting $dV/dz = V'$ and $SC_x/2m = a$ we have

$$\sin\theta = a\rho V/V' \tag{3}$$

Taking account of the third Eq. (1) and Eq. (3) the time in which the vehicle travels the discending path between the altitudes z_i, z_f becomes

$$t = \frac{1}{a}\int_{z_f}^{z_i} \frac{V'}{\rho V^2} dz \tag{4}$$

From the third and fourth Eq. (1), by expressing ctg θ function of $\sin\theta$ as shown in[1] we have

$$x = \int_{z_f}^{z_i} \left(\frac{1}{a\rho V/V'} - \frac{a}{2}\frac{\rho V}{V'} \right) dz \tag{5}$$

The variational problem which confronts us now is to find the minimum of the functional (4) with the condition $x = l$, where l is a given length. As it may be seen, the variational problem is an isoperimetric problem with a moving extremity, since the velocity is not indicated for the final altitude z_f.

The curve which achieves the extremum of the functional (4) is an extremal of the functional

$$J = \int_{z_f}^{z_i} H\, dz \tag{6}$$

where

$$H = V'/\rho V^2 + \Lambda[(1/a)\, V'/\rho V - (a/2)\,\rho V/V'] \tag{7}$$

Λ being a constant.

Taking account of Eq. (7) and of the exponential variation of the density and setting $\Lambda = a\lambda$, Euler's equation leads to the differential equation

$$V'' + \frac{\beta}{2}V' - \frac{V'^2}{V} - \frac{\beta V'^3}{a^2\rho^2 V^2}\left(1 + \frac{1}{\lambda V}\right) = 0 \tag{8}$$

It may be demonstrated with the aid of Legendre and Weierstrass' conditions that the extremal of the problem $V(z)$, solution of Eq. (8), achieves indeed the minimum with constraints of the functional (4).

Equation (8) may be transposed into the system $dV/dz = V'$

$$\frac{dV'}{dz} = -\frac{\beta}{2}V' + \frac{V'^2}{V} + \frac{\beta V'^3}{a^2\rho^2 V^2}\left(1 + \frac{1}{\lambda V}\right) \tag{9}$$

which is numerically integrated as shown in Ref. 1.

With the optimal law of variation of the velocity previously deduced we determine the law variation of the angle θ. Dividing Eqs. (2), considering C_z/C_x = const and integrating with the

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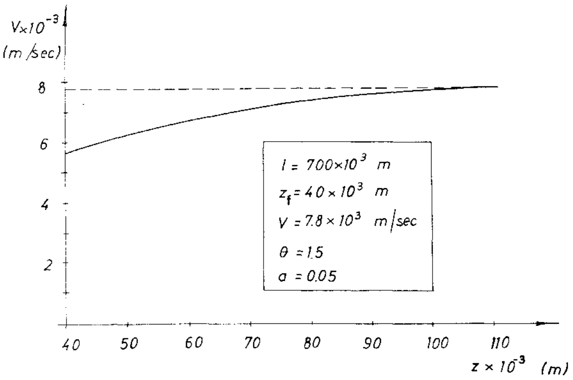

Fig. 1 Optimal variation of the velocity for minimum time entry ($a = 0.05$).

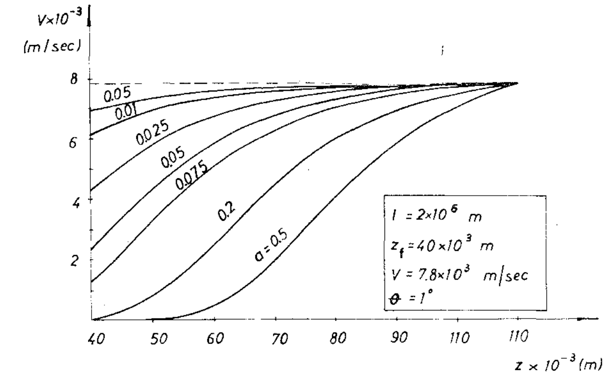

Fig. 3 Optimal variation of the velocity for entry with minimum total heat yielded to the vehicle ($a = 0.005$–0.5).

condition $\theta = \theta_i$ at $V = V_i$ we obtain

$$\theta = (C_z/C_x)\ln(V/V_i) + \theta_i \tag{10}$$

In order to determine the allowable value of the ratio C_z/C_x such that law of variation [Eq. (10)] of the angle θ coincides with the optimal one we use the isoperimetric condition

$$\int_{z_f}^{z_i} \operatorname{ctg}\theta\, dz = l \tag{11}$$

which must be identically satisfied by the angle θ given by Eq. (10). The numerical calculations have shown that in this manner one may determine for all cases of practical interest values of the ratio C_z/C_x which lie within the range of allowable values. In this way the law of variation of the angle θ given by Eq. (10) makes that the arc of trajectory passes at $z = z_f$ through $x = l$ and coincides with the optimal law.

Thus, the arc of optimal trajectory may be numerically calculated with the Runge-Kutta method by the equation

$$dx/dz = -\operatorname{ctg}\theta \tag{12}$$

On the basis of this theory which is applicable both to the entry from space and from the orbit, the graphs of Figs. 1 and 2 give the results of calculations for optimal velocity and a trajectory are for $a = 0.05$ for the case of the entry from the orbit into the Earth's atmosphere (re-entry).

Minimum Total Heat Input Optimal Lifting Entry into Planetary Atmosphere

By expressing the total quantity of heat yielded to the vehicle by the relation

$$Q = S_v \int_{t_i}^{t_f} \left(\frac{d2m}{dt}\right) dt \tag{13}$$

since the mean flux has the expression

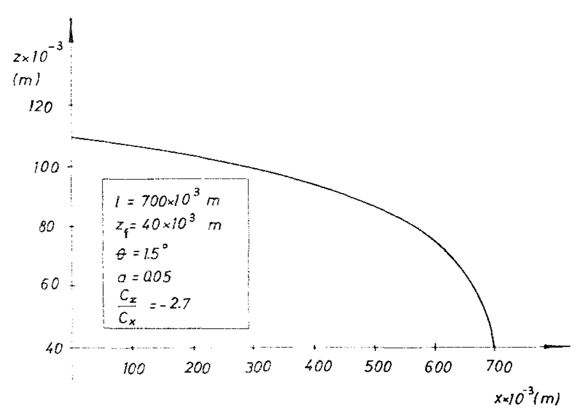

Fig. 2 Optimal trajectory for minimum time entry ($a = 0.05$).

$$d2m/dt = (C_f'/4)\rho V^3 \tag{14}$$

and the equivalent friction coefficient C_f' can be considered approximately constant, we have

$$Q = \frac{S_v C_f'}{4} \int_{t_i}^{t_f} \rho V^3\, dt \tag{15}$$

On introducing the variable z and taking account of the third equation (1) and of Eq. (3) the functional (15) becomes

$$Q = \frac{S_v C_f'}{4a} \int_{z_f}^{z_i} V V'\, dz \tag{16}$$

The problem mentioned, which in this case, too, is an isoperimetric problem with a moving extremity, reduces to the finding of the minimum of the functional (16) with the condition $x = l$.

The curve which achieves the extremum of the functional (16) is an extremal of the functional

$$J = \int_{z_f}^{z_i} H^*\, dz \tag{6'}$$

where

$$H^* = VV' + \lambda^*[(1/a)V'/\rho V - (a/2)\rho V/V'] \tag{7'}$$

λ^* being a constant.

On taking account of Eq. (7′) and of the same exponential law of the density, Euler's equation leads to the equation

$$V'' + \frac{\beta}{2}V' - \frac{V'^2}{V} - \frac{V'^3 \beta e^{2\beta z}}{a^2 \rho_0{}^2 V^2} = 0 \tag{17}$$

whose solution is

$$V = C_2 \exp\left[\int \frac{dz}{\{e^{\beta z}[C_1 - (2/a^2\rho_0{}^2)e^{\beta z}]\}^{1/2}}\right] \tag{18}$$

the two constants C_1 and C_2 being determined from the condition that $z = z_i$, $V = V_i$ and from the isoperimetric condition $x = l$.

In this case, too, it may be demonstrated that the condition of weak minimum of Legendre and that of strong minimum of Weierstrass are satisfied. With the law of optimal variation of the velocity given by Eq. (18) one may deduce in the manner shown previously the law of variation of the angle θ with the aid of which one may calculate the optimal trajectory arc. Calculations of practical interest were carried out and allowable values of the ratio C_z/C_x exist.

Figure 3 is a plot of the results of the calculations carried out for the optimal velocity for various values of a in the case of the entry from the orbit into the Earth atmosphere.

Other Problems

As concerns the other problems previously indicated we mention that for the lifting entry with minimum consumption of ablative mass the variational problem reduces to the finding of the minimum of the functional (K_1 denotes a constant)

$$I = K_1 \int_{z_f}^{z_i} \frac{V'}{(\rho)^{1/2} V} dz \tag{19}$$

with the condition $x = l$. For the lifting entry with minimum heat yielded in the critical zone (denoting by K_2 another constant) one must find the minimum of the functional

$$I^* = K_2 \int_{z_f}^{z_i} \frac{V V'}{(\rho)^{1/2}} dz \tag{20}$$

with the same condition. Both problems may be solved in this manner.

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Laminar Boundary Layer in Noncentered Unsteady Waves

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PREVIOUS analyses[1,2] have considered the laminar wall boundary layer developed within unsteady simple waves which are centered. The present Note concerns the influence of noncentered wave forms which typically occur in practice. A more detailed account is given in Ref. 3.

Figure 1 shows the straight-line mathematical characteristics in the distance-time or x, t plane for inviscid flow generated by simple expansion or compression waves.[4] The wavehead moves at constant sound speed a_0 towards $-x$ into uniform, stationary, perfect gas of constant specific heat ratio γ. The relation $(\gamma-1)u_e/2 + a_e = a_0$ applies locally, and flow properties are constant along the straight characteristics which have slopes $dx/dt = u_e - a_e = u_e/\beta - a_0$ where u = velocity along x, a = sound speed, $\beta = 2/(\gamma+1)$, and subscripts e and 0 denote inviscid flow and gas ahead of the wave, respectively. The first derivatives of the flow properties are assumed to be initially discontinuous, i.e., to be nonzero immediately after the wavehead (cf. Fig. 2). This requires that the wave be generated with nonzero initial acceleration of the gas (say by suitable piston motion or diaphragm rupture). It then follows[3] that the wavehead first derivatives (subscript H) satisfy $(\partial p/\partial t)_H = a_0(\partial p/\partial x)_H = -\gamma\beta p_0/t$ and $(\partial u_e/\partial t)_H = a_0(\partial u_e/\partial x)_H = -\beta a_0/t$. Thus the wavehead first derivatives at any x [say $(\partial p/\partial t)_H$ as measured from Fig. 2] automatically define the corresponding time t and thus uniquely locate the origin of x, t along the wavehead path. The preceding wavehead relations for noncentered waves apply throughout the entire flow within centered waves.

In Fig. 1, e is the x-axis intercept of the straight characteristic through x, t and also represents the x displacement of the local particle at x, t from its position in a centered wave for given velocity u_e. To any point x, t there corresponds a unique value of e determined only by u_e, i.e., $(x-e)/t = u_e/\beta - a_0$. The noncentered flow is thus completely defined by specifying $e(u_e)$. It can be shown[3] that de/du_e as well as e must vanish at the

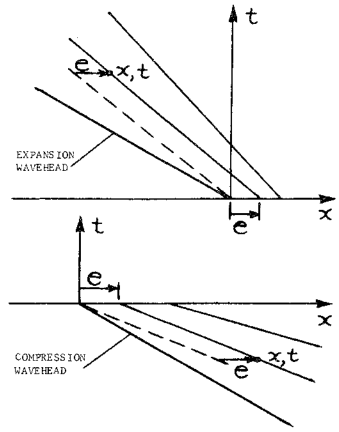

Fig. 1 Distance-time or x, t diagram of straight characteristics for simple noncentered waves.

wavehead. In general, e may be $\gtrless 0$. Thus a constant acceleration piston gives $e > 0$ for compression waves and $e < 0$ for expansion waves. In experiments with simple waves $e(u_e)$ can be determined from a record of $u_e(t)$ or $p_e(t)$ obtained at any x location.[3] For example, the measured pressure-time history of Fig. 2 for a diaphragm-generated expansion wave gives the determination of $e(u_e)$ shown in Fig. 3.

If e is regarded as a function of x and t a new independent variable $s(x, t)$ can be defined as $s = 1 + x/a_0 t - e/a_0 t = s^* - e/a_0 t$. s provides a local conical similarity for the inviscid flow in noncentered waves analogous to s^* for centered waves (where $e \equiv 0$). With centered waves s^* is the ratio of the x distance of any point x, t from the wavehead to the distance of the wavehead from the wave focal point. In terms of s the inviscid flow variables are given by $u_e = \beta a_0 s$ and $(p_e/p_0)^{\varepsilon/\gamma\beta} = (\rho_e/\rho_0)^{\varepsilon/\beta} = (T_e/T_0)^{1/2} = a_e/a_0 = 1 - \varepsilon s$, where ρ is density, T is absolute temperature, and $\varepsilon = (\gamma-1)/(\gamma+1)$. s is > 0 for expansion waves and < 0 for compression waves; its magnitude increases monotonically from zero at the wavehead where $x = -a_0 t$ and $e = 0$. Although the magnitude of $e/a_0 t$ will typically be small, e itself is unrestricted.

Transformation and Solution of Boundary-Layer Equations

The T, ρ, p, and u-velocity fields outside the gas boundary layer are assumed to be those of the noncentered wave discussed. The initial gas boundary layer ($y \geqq 0$, y = distance normal to wall) is described by the customary approximations for two-dimensional, unsteady, compressible laminar boundary layers with heat transfer. Gas viscosity μ is assumed proportional to T; local $\rho\mu$ is thus proportional to $p = p_e(x, t)$. Heat transfer also produces a thin thermal boundary layer in the homo-

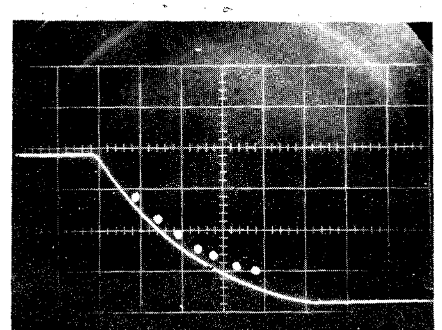

Fig. 2 Oscilloscope trace of sidewall static pressure vs time for expansion wave. 10 psi/cm vertical and 0.5 msec/cm horizontal. Dots indicate centered-wave values. Air with p_0 = 85 psi, T_0 = 74°F. Kistler 603A transducer 3 ft from diaphragm. Tube is $1\frac{1}{2}$ in. × 5 in. cross section. Flow discharges to room through choked 1 in. × 5 in. slot orifice plate at tube end.

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